INTELLIGENCE DIVISION
TELLSPIEL GROUP

ORO-S-650
May 31, 1956

STAFF MEMORANDUM

THE QUEUEIAC PROPORTIONAL SAMPLER

by

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U-18741

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Introduction

In the simulation of queueing networks it is necessary to allow for the possibility that two or more inputs will arrive at a single server, having originated at different sources, with the requirement that they be routed to different destinations. If the simulation is to be simple it is not possible to characterize the input signals such that their identity may be preserved throughout their passage through the network, for this implies very complex electronic circuitry.

In the development of the QUEUEIAC the problem arose: If there are two inputs from distinct sources which must be served, then passed on to different destinations, and if these inputs in reality join the same waiting line, how can the routing destination be maintained while service is performed as though the inputs had joined the same line? This question resulted in the development of the "Proportional Sampler", the function of which is to divide service time between two distinct lines in such a way as to create between the two the same total level of congestion as would have existed had the two inputs joined the same line. This is accomplished by switching service between the two lines, with the time spent on each line being proportional to the instantaneous state of that line.

The present memo summarizes the functional behavior of the QUEUEIAC proportional sampler, and presents the analysis which shows that the proportional sampler indeed performs satisfactorily the purpose for which it was designed.

Operation of the Proportional Sampler

The sampler consists of a double screen-coupled phantastron circuit, which takes a signal from a voltage divider connected to one bank of

contacts of the line stepping switch. The wiper arm thus senses a voltage proportional to the instantaneous state of the line. The service unit is applied to each line a time proportional to the state of that line, switching between the two lines with a frequency: $\omega \sim nm / (n+m)$, where n and m are the states of the two lines. The service spends a length of time serving the first line, which is proportional to n, the length of that line, then switches to the second line, spending a time proportional to m.

Mathematical Analysis of the Proportional Sampler

The switching of the service between the two lines and spending relative amounts of time serving each, proportional to the instantaneous line states, gives rise to coefficients in the difference equation for state probabilities which are functions of the line states. As in Fig. 1 the transition to state (n, m) from the adjacent states above and to the right, involve effective service rates, $\mu_1 + \mu_2$, which depend on n and m.

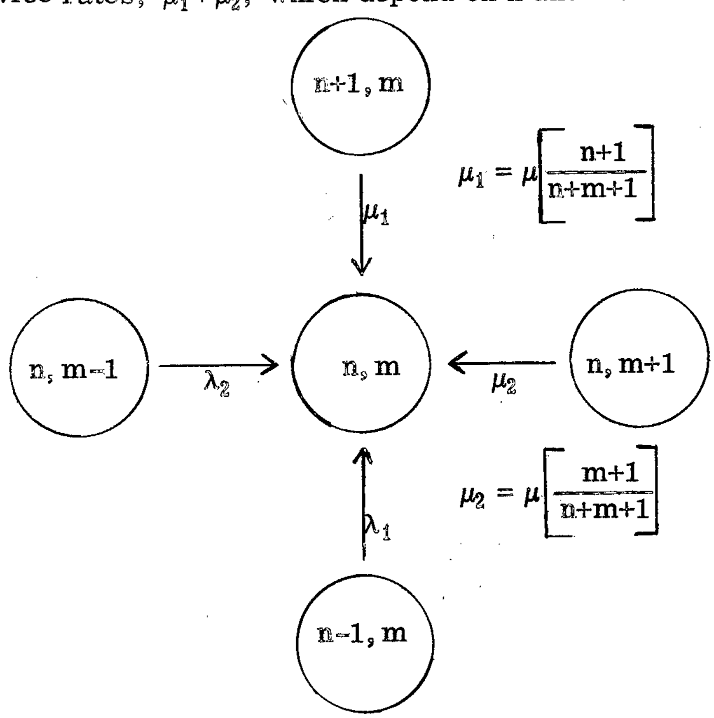

Fig. 1: State Transitions

The equation for a general point of the matrix of state probabilities, and those of the boundaries and origin are:

$$(\lambda+\mu)\,P_n^m = \lambda_1 P_{n-1}^m + \lambda_2 P_n^{m-1} + \frac{n+1}{n+m+1}\,\mu P_{n+1}^m + \frac{m+1}{n+m+1}\,\mu P_n^{m+1} \tag{1}$$

$$(\lambda+\mu)\,P_0^m = \lambda_2 P_0^{m-1} + \mu P_0^{m+1} + \frac{\mu}{m+1} P_1^m \tag{2}$$

$$(\lambda+\mu)\,P_n^0 = \lambda_1 P_{n-1}^0 + \mu P_{n+1}^0 + \frac{\mu}{n+1} P_n^1 \tag{3}$$

$$\lambda P_0^0 = \mu(P_0^1 + P_1^0) \tag{4}$$

This system of difference equations is not easily solved. However, a general solution has been obtained largely by intuition which has permitted the analysis to be concluded *:

$$P_n^m = (1-\rho)\,\rho_1^n\,\rho_2^m\,C_m^{n+m}, \tag{5}$$

where $\rho_1 = \lambda_1/\mu$, $\rho_2 = \lambda_2/\mu$, and C_m^{n+m} are the binomial coefficients.

Proof that the total state of this system, consisting of two lines, is equivalent to a single line classical problem with input rate $\lambda = \lambda_1 + \lambda_2$ and service rate μ is easily obtained. If true, Q_k, the probability of the k'th state (of the classical problem) should be the sum of the P_n^m quantities for which $n + m = k$. That is,

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* This solution was obtained by Lee S. Christie of the Intelligence Division.

$$Q_k = \sum_{j=0}^{k} P_j^{k-j}. \tag{6}$$

Substituting from (5),

$$Q_k = \sum_{j=0}^{k} (1-\rho)\rho_1^j \rho_2^{k-j} \frac{k!}{(k-j)!j!} ,$$

$$Q_k = (1-\rho)\,(\rho_1+\rho_2)^k = \rho^k(1-\rho) \,. \tag{7}$$

By direct calculation it can be shown, for the expected values of n and m, that

$$< n > = \sum_{n,m} nP_n^m = \frac{\rho_1}{1-\rho}, \tag{8}$$

and

$$< m > = \sum_{n,m} mP_n^m = \frac{\rho_2}{1-\rho} \,. \tag{9}$$

From (7), for the total state of the system

$$< s > = \sum_{k=0}^{\infty} kQ_k = \frac{\rho}{1-\rho},$$

so that $< s > = < n > + < m > = < n+m >$. (10)

This is the first conclusion of the analysis that has a direct bearing on the function of the proportional sampler:

The expected total state of the system is precisely the same as that given by the classical problem, if $\lambda_1+\lambda_2 = \lambda$, and the two inputs had joined the same waiting line.

Waiting Times and Times Spent in System

One remaining question requires answer in order that the QUEUEIAC simulation of multiple inputs as described be an adequate representation of the

problem. The handling of two waiting lines as though they were one, requires assurance that the expected time spent in the system is the same, regardless of which line the arriving item joins.

If two inputs of average arrival rates λ_1, and λ_2 were indeed to merge into a single waiting line the expected waiting time and total time in system would be the classical results:

$$W = \frac{\langle n \rangle}{\mu} = \frac{\rho}{\mu(1-\rho)}, \tag{11}$$

and

$$T = \frac{\langle n \rangle + 1}{\mu} = \frac{1}{\mu(1-\rho)} . \tag{12}$$

However, the present problem requires that the two inputs be kept separate, but that they be handled as though they had merged. It is required to find an "effective" service rate μ_1', and likewise a μ_2', which take into account only those states for which $n > 0$. Thus the effective service rate is just the average μ_1, renormalized as indicated.

$$\mu_1' = \frac{\displaystyle\sum_{n,m} \frac{n}{n+m} P_n^m}{1 - \displaystyle\sum_{m=0}^{\infty} P_0^m} \mu, \tag{13}$$

$$= \frac{\mu\rho_1}{1 - \frac{\rho_1}{1-\rho_2}},$$

$$\mu_1' = \mu(1-\rho_2) . \tag{14}$$

Likewise,

$$\mu_2' = \frac{\sum_{n,m} \frac{m}{n+m} P_n^m}{1 - \sum_{n=0}^{\infty} P_n^0} \mu,$$

and

$$\mu_2' = \mu(1-\rho_1) \tag{15}$$

The expected waiting times for the two lines are thus, from (14) and (15):

$$W_1 = \frac{<n>}{\mu_1'} = \frac{<n>}{\mu(1-\rho_2)} = \frac{\rho_1}{\mu(1-\rho_2)\,(1-\rho)}, \tag{16}$$

$$W_2 = \frac{<m>}{\mu_2'} = \frac{<m>}{\mu(1-\rho_1)} = \frac{\rho_2}{\mu(1-\rho_1)\,(1-\rho)}. \tag{17}$$

For times spent in the system,

$$T_1 = \frac{<n>+1}{\mu_1'} = \frac{1}{\mu(1-\rho)}, \tag{18}$$

$$T_2 = \frac{<m>+1}{\mu_2'} = \frac{1}{\mu(1-\rho)}. \tag{19}$$

Thus, $T_1 = T_2$, but $W_1 \neq W_2$. Furthermore, the values for times spent in system are identical with the classical result shown in Eq. (12).

The assertion may be made, therefore that:

The expected time required for the completion of service of an item joining either of two lines, served as described (with the Queueiac Propor-

tional Sampler), is independent of which line the item joins, and is furthermore precisely the same as that expected had the item joined a single queue for which the arrival rate was $\lambda = \lambda_1 + \lambda_2$.

The fact that the waiting times themselves are unequal is to be expected, for the nature of the actual service is peculiar to the proportional sampler. The service is switched back and forth between two items, one in each line, so the knowledge of delay before service is unimportant, for this says nothing about when service will be completed.

Conclusion

It is considered that the analysis presented here is sufficient evidence that the Queueiac Proportional Sampler does indeed accomplish its designed purpose: To handle two parallel inputs as though they had joined the same waiting line, but without actually permitting them to do so. For the expected total state of the system is the same as for a single waiting line with an arrival rate $\lambda = \lambda_1 + \lambda_2$; furthermore, the expected time between arrival and completion of service for an item entering either line is the same as that expected had the item joined a single queue, again for which $\lambda = \lambda_1 + \lambda_2$.